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Algebraic Topology, No. 1
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Sketches of solutions to selected problems

1. Let $A = [0,1], B = [0,1[\,, C =]0,1[$.

   Straightforward computation shows that every interval is homeomorphic to one of A, B, C.

   We need to show that A, B, C are pairwise non-homeomorphic. First, we observe that A is compact, so it is not homeomorphic to B or C, since they are not compact.

   Assume there exists a homeomorphism $f : B \to C$. Then the restriction $f|_{]0,1[} :]0,1[\to]0,1[\setminus f(0)$ is a homeomorphism between a connected and a disconnected topological space, a contradiction.

2. Translate to make the centers agree. Then dilate by a constant factor to make the radii equal. For all of $\mathbb{R}^n$, use some homeomorphism $[0,r[\to [0,\infty[$ instead of the dilation.

3. Wlog ("Without loss of generality"), we assume $0 \in \operatorname{int} C$.

   We give two solutions: one which is more elegant and one which is more direct.

   The direct one first:

   Now we define a function $r : S^{n-1} \to \mathbb{R}^+$ by $r(x) = \max\{\lambda : \lambda x \in C\}$.

   Note that r is bounded and that the maximum exists, because C is compact.

   The map $h : C \to B^n$ defined by

   $$h(x) = \frac{1}{r(x/|x|)} x$$

   if $x \neq 0$ and $h(0) = 0$ can easily be seen to be a homeomorphism with the required properties, once we know that r is continuous.

   Proving that r is continuous: Given a point $x_0 \in S^{n-1}$ and a sequence $x_m \to x_0$ ($x_m \in S^{n-1}$), we need to show that $r(x_m) \to r(x_0)$.

   Let P be the hyperplane orthogonal to x_0 passing through $r(x_0)x_0$.

   Choose n points $y_i' \in P$ $(1 \leq i \leq n)$ such that they are affinely independent and:

   $$r(x_0)x_0 \in \operatorname{int} \langle y_1', \ldots, y_n' \rangle$$

   Let $y_i := y_i'/|y_i'|$. By convexity, the simplex with vertices $r(y_1)y_1, \ldots, r(y_n)y_n$, $r(x_0)x_0$ (which are not necessarily affinely independent) is contained in C, showing that $\liminf r(x_m) \geq r(x_0)$.

On the other hand, we can easily extract a subsequence x_{m_i} such that $r(x_{m_i}) \to \limsup r(x_m)$. Because C is closed, the sequence $r(x_{m_i})x_{m_i}$ converges to a point in C, which shows $\limsup r(x_m) \leq r(x_0)$.

Now the elegant solution:

Define the Minkowski functional $p\colon \mathbb{R}^n \to \mathbb{R}_+$ by

$$p(x) := \mu_C(x) = \inf\{\lambda > 0 : \frac{1}{\lambda}x \in C\} = \inf\{\lambda > 0 : x \in \lambda C\}$$

Then p satisfies $p(0) = 0$, $p(\alpha x) = \alpha p(x)$ for $\alpha \geq 0$, and $p(x + y) \leq p(x) + p(y)$. You can check this yourself; alternatively, you can read a proof in Rudin, *Functional Analysis*, 1.35.

There are $m > 0, M > 0$ such that $m\|x\| \leq p(x) \leq M\|x\|$ for all x.

It follows easily that p is continuous and that $f\colon \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$f(x) = \frac{p(x)}{||x||}x$$

is continuous (also in 0) and maps C homeomorphically onto $\mathcal{B}_1$.

6. The surface given by *abab* is the projective plane. To prove that *aabb* is the Klein bottle $yzy^{-1}z$, one needs three steps of cutting and gluing (you can find an illustration in Tom Dieck, *Topologie*, II.4, Theorem 4.5 and 4.8).

8. The sphere S^2 is the neutral element for connected sums of surfaces (when you remove a disk, the remainder is a disk). When you glue two surfaces, and at least one of them is non-orientable, then the result will be non-orientable. If X is a non-orientable surface, then $X\#T \simeq X\#K$. You can read this up in Tom Dieck, *Topologie*, II.4, as well.