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Algebraic Topology, No. 2
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Sketches of solutions to selected problems

1. First, we observe that $H_0(X;\mathbb{Z}_2) = \mathbb{Z}_2$ for all connected complexes X: We have

$$C_0 = Z_0 = \mathcal{P}(X_0)$$

because ∂ maps all of C_0 to 0 (since we have $\{0\} = C_{-1}$). Here, we denote by X_0 the set of points (0-dimensional simplices).

We have $B_0 = \{A \in \mathcal{P}(X_0) | \, |A| \text{ is even}\}$, so $H_1 = \{[\emptyset], [\{a\}]\}$ for some point $a \in X_0$.

For $K_{3,3}$, denote the vertices by $t_1, t_2, t_3, b_1, b_2, b_3$, such that every t-vertex is connected with every b-vertex by an edge.

Since $\mathbb{Z}_2$ is a field, H_1 needs to be a $\mathbb{Z}_2$-vectorspace. Hence, it suffices to find a maximal set of linearly independent 1-cycles (i.e.: circles in the graph).

Choose the circles t_1, b_2, t_2, b_1, t_1 and t_2, b_3, t_3, b_2, t_2 and t_1, b_3, t_3, b_1, t_1 and t_1, b_3, t_3, b_2, t_1.

Convince yourself that these circles are linearly independent and that every other circle can be written as a sum of the circles we have given (you can find a proof in next week's model solutions, where the homology groups over $\mathbb{Z}$ of the same graph are calculated).

Hence $H_1(K_{3,3};\mathbb{Z}_2) = \mathbb{Z}_2^4$.

2. First, we clearly have $H_0(T;\mathbb{Z}_2) = \mathbb{Z}_2$ again, since T is connected.

For calculating $H_2(T;\mathbb{Z}_2)$, we try to find the 2-cycles: Start with any 2-simplex in the triangulation of T. In order to obtain a 2-cycle, i.e. a 2-chain without boundary, you have to take the set of all 2-simplices (you need to add another one and another one ... in order to avoid having boundary; so you end up with all of them). Note that T satisfies $\partial T = 0$, because every 1-simplex occurs twice in its boundary. Hence, we have $H_2(T;\mathbb{Z}_2) = \{\emptyset, T\} = \mathbb{Z}_2$.

We want to show $H_2(T;\mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

To that end, we choose the following two 1-chains, and prove that they are a basis: We pick the lower horizontal line, and call it a and we let b denote the vertical left line (note that they are both 1-cycles).

We show that a and b are linearly independent:

If you try to find a 2-chain which has $a - b = a + b$ as boundary, you start with some 2-simplex adjacent to some part of a or b. Similar to the case when we were looking

for H_2, we notice that we have to take all the 2-simplices, because otherwise, some interior boundary elements would also remain as a boundary. Since $\partial T = 0 \neq a + b$, this shows that $[a] \neq [b]$, so $\mathbb{Z}_2[a] \oplus \mathbb{Z}_2[b] \leq H_1$. Convince yourself that every other 1-cycle is homologous to one of $\emptyset, a, b, a + b$!

The Klein bottle has the same homology groups with coefficients in $\mathbb{Z}_2$ as the torus (you can do the same calculation as before). Intuitively, this is because the group $\mathbb{Z}_2$ does not take orientations into account. We will soon introduce homology with coefficients in $\mathbb{Z}$. This theory will be powerful enough to show that there is a difference between T and K.

3. The given simplicial complex (indeed, it is a simplicial complex) triangulates the torus T. Just glue the left and right line first. You obtain a cylinder, and you glue the boundary with the same orientation as in the standard torus, but with a little "twist".

   The Möbius triangulation is actually not the one given on the problem sheet, but the one where the upper row looks as follows: "4 5 6 7 1 2 3 4". This triangulation of the torus has the interesting property that it can be realized in $\mathbb{R}^3$ (with line segments and triangles, without bending).

4. The projective plane P satisfies $H_0(P; \mathbb{Z}_2) = H_1(P; \mathbb{Z}_2) = H_2(P; \mathbb{Z}_2) = \mathbb{Z}_2$.

   Probably the easiest triangulation of the projective plane is the following:
   Start with the square (which we want to glue via $abab$), divide it into four subsquares, and divide each of these small squares into four triangles (by adding one vertex in the middle).

5. Both the strip and the Möbius band satisfy $H_0 = H_1 = \mathbb{Z}_2$ (with coefficents in $\mathbb{Z}_2$).