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Algebraic Topology, No. 3
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Sketches of solutions to selected problems

2. a) For $K_{3,3}$, denote the vertices by $t_1, t_2, t_3, b_1, b_2, b_3$, such that every t-vertex is connected with every b-vertex by an edge.

   First, we observe that $H_0(K_{3,3}) = \mathbb{Z}$ (by 4.5.1).

   For H_1, consider the four circles $a := (t_1, b_2, t_2, b_1, t_1)$, $b := (t_2, b_3, t_3, b_2, t_2)$, $c := (t_1, b_3, t_3, b_1, t_1)$ and $d := (t_1, b_3, t_3, b_2, t_1)$.

   Let us show that these circles are linearly independent: If we have some linear combination of them which yields 0, then the coefficients of the first three circles need to be 0, because every one of them contains an edge which occurs in this circle only ($(t_2, b_1), (t_2, b_3), (t_3, b_1)$ resp.). Hence, the coefficient of d needs to be 0 as well.

   In order to show that every 1-cycle z (i.e. set of circles with coefficients) is homologous to a linear combination of the four given circles $z \sim \alpha a + \beta b + \gamma c + \delta d$, note that we have no choice for the coefficients of a, b, c (by the property above: these three circles contain an edge which is not contained in one of the other basis circles). So $z - \alpha a - \beta b - \gamma c$ is a 1-cycle in the graph with the three arcs $(t_2, b_1), (t_2, b_3), (t_3, b_1)$ removed. But (draw a sketch!) in this graph, every 1-cycle is homologous to a multiple of d, since this is the only non-trivial circle in the remaining graph.

   Hence $H_1(K_{3,3}; \mathbb{Z}) = \mathbb{Z}^4$ (see 4.5.2 as well).

   c) We observe in a similar way to exercise 2.2.b, that the only possible nontrivial elements of $H_2(K)$ need to be multiples of the chain where all the two-simplices have equal coefficients. However, this chain does not have boundary 0. Instead, we have $\partial K = aba^{-1}b = bb$. Hence, we have $H_2(K) = 0$, and at the same time, this is the only relevant relation of the two 1-cycles a and b (defined as in 2.2). Hence we have $H_1(K) = \mathbb{Z} \oplus \mathbb{Z}_2$ (see 4.5.4).

4. a) The homology groups of the projective plane P are

$$H_0(P) = \mathbb{Z}, H_1(P) = \mathbb{Z}_2, H_2(P) = 0$$

   (Note: For a non-orientable closed surface, there is always an element of order 2 in H_1, and H_2 is always 0.)

   b) The connected sum of two tori has the symbol $aba^{-1}b^{-1}cdc^{-1}d^{-1}$. Since the homology group is commutative, this shows (by arguments similar to 2.2) that $H_2(T\#T) = \mathbb{Z}$. Furthermore, one sees that the cycles a, b, c, d generate H_1 and are linearly independent, so $H_1(T\#T) = \mathbb{Z}^4$.