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Algebraic Topology, No. 8
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Sketches of solutions to selected problems

1. a) To obtain $f \approx f$, define the homotopy $F_t(x) = f(x)$.
To obtain $g \approx f$ under the assumption $f \approx g$: Denote the given homotopy by F_t (with $F_0 = f$ and $F_1 = g$). Then $G_t(x) = F_{1-t}(x)$ is a homotopy which shows $g \approx f$.
Now let $f \approx g$ and $g \approx h$ with given homotopies F, G resp. Define

$$H_t(x) := \begin{cases} F_{2t}(x) & t \in [0, 1/2] \\ G_{2t-1}(x) & t > 1/2 \end{cases}$$

Now H is continuous (because $F_1 = G_0$), and satisfies $H_0 = f$, and $H_1 = h$.

b) We are given a homotopy F_t (from $f \approx g$) and a homotopy G_t (from $f' \approx g'$). Define $H_t(x) = G_t(F_t(x))$. Now, being a composition of continuous functions, H is continuous, and satisfies $H_0 = f' \circ f$ and $H_1 = g' \circ g$.

2. a) Let $f : S^n \to S^n$ be a continuous function which is not surjective. Pick a point $\infty \notin f(S^n)$ and use stereographic projection $h : S^n \backslash \{\infty\} \to R^n$ (which is a homeomorphism) (for information on stereographic projection, you can compare exercise 14e from last semester's lecture on differential geometry by Große-Brauckmann, for example; the lecture notes and exercise sheets are available from his homepage).
Now $h \circ f$ is nullhomotopic (because $\mathbb{R}^n$ is contractible by the homotopy $F_t(x) = tx$). This implies that f is nullhomotopic (by the homotopy $F'_t(x) = h^{-1}[thf(x)]$).

b) The map $\mathrm{id} : S^n \to S^n$ is not nullhomotopic. By 6.3.4, $\mathrm{id}_* = 0$ would have to hold if id was nullhomotopic. However, this is not true for the induced map on the n-th homology group: $H_n(S^n) = \mathbb{Z}$ and $\mathrm{id}_* = id \neq 0$ in this dimension.

3. a) For the strip $S^1 \times [0, 1]$, we can use the homotopy $F_t(x, a) = (x, ta)$ to obtain $F_1 = \mathrm{id}$ and $F_0 = S^1 \times \{0\} \simeq S^1$. Since we have $F_t(x, 0) = (x, 0)$, the space S_1 is a deformation retract of $S^1 \times [0, 1]$.

b) We can view the Möbius band M as the space obtained from $[-1, 1] \times [-1, 1]$ by gluing the "left end" to the "right end" with a twist. Formally, the identification is $(-1, a) \sim (1, -a)$.
On this space, we define $F_t(x, a) = (x, ta)$. We need to check that F is well defined on $M = ([-1, 1] \times [-1, 1]) / \sim$. This is the case, because $F_t(-1, a) = (-1, ta) \sim (1, -ta) = F_t(1, -a)$.
Observe that we have $S^1 \simeq F_0(M)$, since $F_0(M) = [-1, 1] \times \{0\}$ with the identification $(-1, 0) \sim (1, 0)$.
This shows that $S^1 \simeq F_0(M)$ is a deformation retract of M, because $F_1 = \mathrm{id}$.

5. a) The boundary A of the strip X consists of two disjoint circles. By exercise 7.5, we have

$$H_q(A) = \begin{cases} \mathbb{Z}^2 & q \in \{0,1\} \\ 0 & \text{otherwise} \end{cases}$$

By exercise 8.3.a, we know the homology groups of the strip to be the same as the homology groups of the circle S^1.

The long exact sequence shows $0 = H_q(X) = H_q(X, A)$ if $q \geq 3$.

The rest of the long exact sequence looks as follows:

$$0 \longleftarrow H_0(X,A) \longleftarrow \underbrace{H_0(X)}_{\mathbb{Z}} \xleftarrow{\alpha} \underbrace{H_0(A)}_{\mathbb{Z}^2} \xleftarrow{\beta} H_1(X,A) \xleftarrow{\gamma} \underbrace{H_1(X)}_{\mathbb{Z}} \xleftarrow{\delta} \underbrace{H_1(A)}_{\mathbb{Z}^2}$$
$$\xleftarrow{\varepsilon} H_2(X,A) \longleftarrow \underbrace{H_2(X)}_{0}$$

First, we note that α is clearly surjective, so $H_0(X, A) = 0$.

Next, we consider the map δ. Observe that $H_1(A)$ is generated by the two circles. If we choose the same orientation on the two circles and identify $H_1(A)$ with $\mathbb{Z}^2$ (by counting how many times the upper circle, resp. the lower circle is used), we see that $\delta|_{(\bullet,0)}$ is an isomorphism, and $\delta|_{\mathbb{Z}(1,-1)} \equiv 0$.

Since ε is injective, we have obtained $H_2(X, A) \cong \mathbb{Z}$.

Further, δ being surjective implies β being injective, and by the same argument as above, we obtain $H_1(X, A) \cong \mathbb{Z}$.

b) For the Möbius band, A is one circle only. For $q \geq 3$, we still have $0 = H_q(X) = H_q(X, A)$. Also, $H_0(X, A) = 0$ still holds (actually, we have shown that $H_0(X, A) = 0$ whenever A is not empty at some point).

We consider the same diagram as above, with the difference that $H_0(A) = H_1(A) = \mathbb{Z}$.

We need to understand δ. Since we know that the deformation retraction from 8.3.b induces an isomorphism in homology, and $\delta = i_*$ for the inclusion $A \to X$, we consider the image of the circle generating $H_1(A) = \mathbb{Z}$ under $F_0 \circ i$. This image is twice the circle generating $H_1(X)$. Hence, δ is just multiplication by two. This implies $H_2(X, A) \cong \mathrm{im}(\varepsilon) = \ker(\delta) = 0$.

Since α is an isomorphism, we have $\ker(\beta) = H_1(X, A) = \mathrm{im}(\gamma)$. From our knowledge of δ, we obtain $H_1(X, A) = \mathbb{Z}_2$.

7. Considering the long exact sequence in homology, $H_q(A) \cong H_{q+1}(X, A)$ for $q > 0$ is immediate because $H_q(X) = 0$ for $q > 0$ (by 6.2.7).

To obtain information on $H_1(X, A)$, we first note that we have $H_0(X, A) = 0$ as before. Hence, the corresponding part of the long exact sequence in homology is a short exact sequence. This short exact sequence splits, because we can choose an element of A as generator of $H_0(X) \cong \mathbb{Z}$. Therefore, $H_0(A) \cong H_0(X) \oplus H_1(X, A) = \mathbb{Z} \oplus H_1(X, A)$.