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Sketches of solutions to selected problems

b) The projection $p(x_0, \ldots, x_n) = (x_0, \ldots, x_{n-1})$, when restricted to S^n_+ or S^n_-, yields a homeomorphism from $S^n_\pm$ to B^n.

c) Choose $B := \{\text{South Pole}\}$, and $A := S^n_-$. Using excision, we obtain $H_q(S^n, S^n_-) \cong H_q(S^n \backslash B, S^n_- \backslash B)$.

The pairs of spaces $(S^n \backslash B, S^n_- \backslash B)$ and (S^n_+, S^{n-1}) are clearly homotopy equivalent (in the sense one derives from 6.3.3).

Hence, the result follows from homotopy invariance.

d) First, we note that $H_0(S^0) = \mathbb{Z}^2$ and $H_q(S^0) = 0$ if $q \neq 0$ (by exercises 7.1 and 7.5).

Next, we observe $H_0(S^n) = \mathbb{Z}$ if $n \neq 0$ because S^n is connected.

For $q > 1$, we get $H_{q-1}(S^{n-1}) \cong H_q(S^n_+, S^{n-1})$ from the long exact sequence of the pair (S^n_+, S^{n-1}) and $H_q(S^n, S^n_-) \cong H_q(S^n)$ from the long exact sequence of the pair (S^n, S^n_-) (because $S^n_\pm$ is contractible).

Using part (c), we conclude $H_{q-1}(S^{n-1}) \cong H_q(S^n)$.

Thus we are done if we can compute $H_1(S^n)$ to be what we expect them to be.

From the long exact sequence of (S^1_+, S^0), we obtain $H_1(S^1_+, S^0) = \mathbb{Z}$, and for the pair (S^1, S^1_-), we obtain $H_1(S^1) \cong H_1(S^1, S^1_-) \cong \mathbb{Z}$ (the last isomorphism comes from part (c)).

Similarly, we obtain $H_1(S^n) = 0$ for $n > 1$.

e) This follows immediately from the previous parts.

f) S^{n-1} is a deformation retract of $\mathbb{R}^n \backslash \{0\}$ and $B^n \backslash \{0\}$.

g) This follows immediately from the long exact sequence of the respective pairs (which contains a lot of 0's).

h) If S^n and S^m are homotopy equivalent (or homeomorphic), all of their homology groups have to be isomorphic. Hence, $n = m$ is the only possibility for that to happen.

i) Compare exercise 7.III.

Observe that $\mathbb{R}^n$ is homotopy equivalent to $\mathbb{R}^0$ (a point) since it is contractible. Hence, $\mathbb{R}^n$ is also homotopy equivalent to $\mathbb{R}^m$ for arbitrary choice of n, m.

(Explicitely: for $n < m$, the standard projection(p)/inclusion(i) maps are homotopy inverses of each other: Indeed, $p \circ i = id$ and $i \circ p \simeq id$ via $F_t(x^n, x^{m-n}) = (x^n, tx^{m-n})$.