A short introduction to Topological Vector Spaces

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These notes just present the definitions, some examples/exercises, and the theorems without proofs. They are heavily based on W. Rudin, "Functional Analysis", published by McGraw-Hill, Inc.

1 General

We will start off by giving some definitions:

A vector space shall always be a vector space over $\mathbb{R}$ or $\mathbb{C}$; we will use $\mathbb{K}$ to stand for either one of them.

Definition 1.1: For two subsets A, B of a vector space, $\lambda \in \mathbb{K}$

a) $$A + B := \{a + b : a \in A, b \in B\}$$

b) $$\lambda A := \{\lambda a : a \in A\}$$

Exercise 1.1: Find a set of real numbers such that $2A \neq A + A$.

Definition 1.2: A subset C of a vector space is *convex* if

$$a, b \in C, t \in [0, 1] \Rightarrow ta + (1-t)b \in C$$

or equivalently

$$tC + (1-t)C \subset C \qquad (0 \leq t \leq 1)$$

Exercise 1.2: Show that if a subset A of a vector space is convex and contains 0, then

$$\underbrace{A + A + \cdots + A}_{n\text{-times}} = nA$$

Definition 1.3: A subset B of a vector space is *balanced* if $\alpha B \subset B$ for every $\alpha \in \mathbb{K}$ with $|\alpha| \leq 1$.

Exercise 1.3: Find a subset of $\mathbb{R}^2$ which is balanced, but fails to be balanced if it is considered to be a subset of $\mathbb{C}$.

Exercise 1.4: Show that for a balanced set B, and some numbers $n_1 < n_2 < \cdots < n_s$,

$$\bigcup_{i=1}^{s} n_i B = n_s B$$

while for a non-balanced set, this equality does not necessarily hold.

Definition 1.4: In a topological space, a local base at a point a is a collection $\mathcal{B}$ of open sets containing a such that every open set containing a contains some element of $\mathcal{B}$.

Definition 1.5: A *topological vector space (TVS)* is a vector space X with a topology such that addition and scalar multiplication are continuous and for all $x \in X$, the set $\{x\}$ is closed.

Remark: Not every author requires sets of the form $\{x\}$ to be closed. But then they need this as an assumption to most of the theorems. I stick with Rudin's convention to include it in the definition.

I will use the expression $\mathcal{N}_0$ as the set of all neighborhoods of 0.

Here are some motivating examples of TVSs:

Exercise 1.5: a) Show that a normed vector space is a TVS (with the topology induced by the norm).

b) Conclude that $\mathbb{R}^n, l^p, L^p$ are TVSs.

c) Conclude that every Banach or Hilbert space is a TVS.

d) Use the above to show that $\mathbb{R}^\infty$ with the product topology as in Lawson, "A geometric introduction to topology", 1.9.64 is a TVS.

Also, $C(\Omega)$, the set of continuous (complex-valued) functions on some open subset Ω of $\mathbb{R}^n$ has a natural topology, which makes it into a TVS. However, this is not a normed space (in fact, one can prove that there is no norm on this set that induces the topology).

From now on, let X be a TVS.

Definition 1.6: A subset B of X is *bounded* if for every neighborhood V of 0, there exists a number s such that $t > s \Rightarrow B \subset tV$.

Exercise 1.6: Check that for a normed vector space, this definition of boundedness agrees with the usual definition in terms of the norm.

Remark: For an arbitrary metric space, this definition of boundedness needs not coincide with the usual one for metric spaces! (Just consider a metric space where the metric is bounded).

Definition 1.7: For an element a of X, define the *translation operator* $T_a(x) = a + x$, and for $\lambda \in \mathbb{K}\backslash\{0\}$, define the *multiplication operator* $M_\lambda(x) = \lambda x$.

Proposition 1.1. *T_a and M_λ are homeomorphisms of X.*

This proposition shows in particular that the topology of X is *translation-invariant* or simply *invariant* in the following sense: $A \subset X$ is open iff $a + A$ is open for all $a \in X$. Thus the topology is completely determined by any local base. Therefore, when talking about a local base of a TVS we always mean a local base at 0.

Definition 1.8 (Types of topological Vector spaces): a) X is *locally convex* if there is a local base whose members are convex.

b) X is *locally bounded* if 0 has a bounded neighborhood.

c) X is *locally compact* if 0 has a neighborhood whose closure is compact.

Exercise 1.7: Show that a normed space is locally convex and locally bounded.

Lemma 1.2. *Let W be a neighborhood of 0 in X. There is a neighborhood U of 0 which satisfies $U = -U$ (i.e. U is symmetric) and $U + U \subset W$.*

Corollary 1.3. *Let W be a neighborhood of 0 in X. For every $n \in \mathbb{N}$, there exists a symmetric neighborhood U of 0 such that*

$$\underbrace{U + U + \cdots + U}_{n\text{-times}} \subset W$$

Theorem 1.4. *Let K be a compact subset of X, and C be a closed subset of X which does not intersect K. Then there is a neighborhood V of 0 such that*

$$(K + V) \cap (C + V) = \emptyset$$

Corollary 1.5. *Every TVS is a regular Hausdorff space.*

Corollary 1.6. *Let $\mathcal{B}$ be a local base for X. Then every member of $\mathcal{B}$ contains the closure of some member of $\mathcal{B}$.*

Theorem 1.7. *a) For $A \subset X$, we have $\overline{A} = \bigcap(A + V)$, where V runs through all neighborhoods of 0.*

b) Let $A, B \subset X$. Then $\overline{A} + \overline{B} \subset \overline{A + B}$.

c) If Y is a subspace of X, so is $\overline{Y}$.

d) If C is a convex subset of X, so are $\overline{C}$ and C°.

e) If B is a balanced subset of X, so is $\overline{B}$. If $0 \in B^\circ$, then B° is balanced.

f) If E is a bounded subset of X, so are $\overline{E}$ and E°.

The inclusion in (b) may be strict. The following exercise tells a little more:

Exercise 1.8: a) Show that A compact, B closed implies that $A + B$ is closed.

b) Find two closed subsets A, B of $\mathbb{R}^2$ such that $A + B$ is not closed.

Theorem 1.8. *a) Every neighborhood of 0 contains a balanced neighborhood of 0.*

b) Every convex neighborhood of 0 contains a balanced convex neighborhood of 0.

Corollary 1.9. *a) Every topological vector space has a balanced local base.*

b) Every locally convex TVS has a balanced convex local base.

Theorem 1.10. *Suppose V is a neighborhood of 0 in X, and $(r_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence of positive real numbers that tends to infinity as n goes to infinity.*

a) $X = \bigcup_{n=1}^{\infty} r_n V$

b) Every compact subset K of X is bounded.

c) If V is bounded, then the collection

$$\left\{ \frac{1}{r_n} V : n \in \mathbb{N} \right\}$$

is a local base.

Remark: This shows in particular that a locally bounded space has a countable local base (which will imply some other facts like in Theorems 2.4, 3.1)

2 Linear Mappings

If Λ is a linear map, one often writes Λx instead of $\Lambda(x)$.

Exercise 2.1: Let A be a subset of X and B be a subset of Y, some other TVS, and Λ be a linear map from X to Y.

a) If A is a subspace (or a convex set, or a balanced set), then so is $\Lambda(A)$.

b) If B is a subspace (or a convex set, or a balanced set), then so is $\Lambda^{-1}(B)$.

From this exercise, note in particular that $\Lambda^{-1}(\{0\}) = \ker \Lambda$, the *kernel* or *null space* of Λ is a subspace of X. Rudin calls the kernel $\mathcal{N}(\Lambda)$.

From now on, Λ will always stand for a linear map between TVSs X and Y.

Definition 2.1: Λ is *bounded* if it maps every bounded set to a bounded set.

Exercise 2.2: Check that for a linear map between normed spaces this definition agrees with the one usually applied in this case.

Theorem 2.1. *The following are equivalent:*

- Λ *is continuous.*
- Λ *is continuous at 0.*

In particular, if one of the above holds, then Λ *is uniformly continuous in the following sense: For each neighborhood* W *of 0 in* Y*, there is a neighborhood* V *of 0 in* X *such that* $y - x \in V \Rightarrow \Lambda y - \Lambda x \in W$.

Theorem 2.2. *If* A *is a linear functional (i.e. a linear map into* $\mathbb{K}$*), which is not identically 0, then the following are equivalent:*

a) A *is continuous.*

b) $\ker A$ *is closed.*

c) $\ker A$ *is not dense in* X.

d) A *is bounded in some neighborhood* V *of 0.*

For the proof of the last implication of the next theorem, we need the following

Lemma 2.3. *If* Λ *is not continuous, then there exists a neighborhood* W *of 0 in* Y *such that for every neighborhood* V *of 0 in* X *and every* $n \in \mathbb{N}$*, there exists* $x \in V$ *such that* $\Lambda x \notin nW$.

Theorem 2.4. *We have the following chain of implications (where* $(x_n)_n$ *is a sequence converging to 0 in* X*):*

$$\Lambda \textit{ is continuous} \Rightarrow \Lambda \textit{ is bounded} \Rightarrow \{\Lambda x_n\} \textit{ is bounded}$$

If X *has a countable local base, then the three statements above are equivalant.*

Exercise 2.3: Show (using Theorem 2.4) that if X has a countable local base, then for a linear map Λ it is equivalent to be continuous and to have the following property: For every sequence $(x_n)_n$ converging to 0 in X, the sequence $(\Lambda x_n)_n$ converges to 0 (in Y).

Exercise 2.4: Generalize the previous exercise to show that it suffices to have: for every convergent sequence $(x_n)_n$, the sequence $(\Lambda x_n)_n$ converges to $\Lambda\big(\lim_{n\to\infty} x_n\big)$

3 Metrizability

Definition 3.1: Let d be a metric on X.

a) d is called *(translation-)invariant* if $d(x, y) = d(x + z, y + z)$ for all $x, y, z \in X$.

b) If the topology induced by d is the same as the (given) topology on X, then d is said to be *compatible* with the topology of X.

Theorem 3.1. If X has a countable local base, then X is metrizable.
In particular, there exists a metric d on X which satisfies:

- *d is compatible with the topology of X.*
- *d is invariant.*
- *The open balls centered at 0 are balanced.*
- *If X is locally convex, we can choose the metric such that the open balls are convex.*

Remark: Note that here, a conflict regarding boundedness might arise: The d-bounded sets do not necessarily agree with the bounded sets according to definition 1.6. In particular, the metric constructed in the proof of the preceding theorem makes the whole space bounded ($d(x, y) \leq 1$), which is not the case using the topologic definition for any nontrivial TVS.

Definition 3.2 (more types of TVSs): a) X is an *F-space* if its topology is induced by a complete, invariant metric.

b) X is a *Fréchet space* if it is a locally convex F-space.

Remark: The terminology above is (once again) depending on the author. There are some texts which use exactly the opposite terminology, and texts which do not make the distinction between the two.

4 Locally compact spaces, finite-dimensional spaces

In this section, the main result will be that the locally compact TVSs are exactly the finite-dimensional ones, and the latter ones are homeomorphic to $\mathbb{K}^n$.

Lemma 4.1. *If $f : \mathbb{K}^n \to X$ is linear, then f is continuous.*

Theorem 4.2. *If Y is an n-dimensional subspace of X, then:*

a) Every isomorphism of $\mathbb{K}^n$ onto Y is a homeomorphism.

b) Y is closed.

Theorem 4.3. *Every locally compact TVS X has finite dimension.*

We say that X has the *Heine-Borel property*, if every closed and bounded subset is compact (note that every compact subset is closed and bounded by Theorems 1.5, 1.10).

Exercise 4.1: Show: If X is locally bounded and has the Heine-Borel property, then it is finite dimensional.

This fact is most often used in the following context: If X is an infinite-dimensional space, and we can show that it has the Heine-Borel property, then it cannot be locally bounded.

5 Seminorms and Local Convexity

The goal of this section to show the one-to-one correspondence between locally convex TVSs and TVSs whose topology is induced by a family of separating seminorms. We need some new terminology first.

Definition 5.1: A *seminorm* is a function $p : X \to \mathbb{R}$ such that

- $p(x + y) \leq p(x) + p(y)$
- $p(\alpha x) = |\alpha|\, p(x)$ for all $\alpha \in \mathbb{K}$.

Definition 5.2: A family of seminorms on X is said to be *separating* if for every $x \neq 0$, there is at least one seminorm p in the family such that $p(x) \neq 0$.

Definition 5.3: A convex set A is called *absorbing* if for every $x \in X$, there is a positive real number t such that $x \in tA$ (or equivalently $t^{-1}x \in A$).

Definition 5.4: For an absorbing set A, the *Minkowski functional* $\mu_A : X \to \mathbb{R}$ is defined by

$$\mu_A(x) = \inf\{t > 0 : x \in tA\} = \inf\{t > 0 : t^{-1}x \in A\}$$

Theorem 5.1. *For a seminorm p on X, we have:*

a) $p(0) = 0$

b) $p(x) \geq 0$

c) $|p(x) - p(y)| \leq p(x - y)$

d) $\{x : p(x) = 0\}$ is a subspace of X.

e) The set $B = \{x : p(x) < 1\}$ is convex, balanced, absorbing, and $p = \mu_B$.

Remark: The last theorem shows that a seminorm is a norm if $p(x) = 0 \Leftrightarrow x = 0$.

Theorem 5.2. *If B is a convex absorbing set in X, then*

a) $\mu_B(x+y) \leq \mu_B(x) + \mu_B(y)$

b) $\mu_B(tx) = t\mu_B(x)$ *if* $t \geq 0$

c) μ_B *is a seminorm if B is balanced.*

d) If $A = \{x : \mu_B(x) < 1\}$ *and* $C = \{x : \mu_B(x) \leq 1\}$*, then* $A \subset B \subset C$ *and* $\mu_A = \mu_B = \mu_C$.

Theorem 5.3. *Let X be locally convex and $\mathcal{B}$ be a convex balanced local base.*

a) $V = \{x \in X : \mu_V(x) < 1\}$ *for every* $V \in \mathcal{B}$.

b) $\{\mu_V : V \in B\}$ *is a separating family of continuous seminorms on X.*

Theorem 5.4. *Let $\mathcal{P}$ be a separating family of seminorms on a vector space X. For each $p \in \mathcal{P}$ and $n \in \mathbb{N}$, let*

$$V(p,n) = \{x : p(x) < 1/n\}$$

Let $\mathcal{B}$ be the collection of all finite intersections of sets of the form $V(p,n)$. Then $\mathcal{B}$ is a convex balanced local base for a topology on X, which turns X into a locally convex TVS such that

a) Every $p \in \mathcal{P}$ is continuous.

b) A set $E \subset X$ is bounded if and only if for every $p \in \mathcal{P}$, the set $p(E)$ is bounded.

Exercise 5.1: If X is a locally convex TVS, and $\mathcal{B}$ is a convex balanced local base, show that the topology τ on X is equal to the topology τ' generated by the procedure from theorem 5.4 (where the seminorms arise from $\mathcal{B}$ by theorem 5.3).

Exercise 5.2: If $\{p_1, p_2, \dots\}$ is a countable separating family of seminorms, show that

$$d(x,y) = \max_i 2^{-i} \frac{p_i(x-y)}{1+p_i(x-y)}$$

defines an invariant metric on X, which is compatible to the topology induced by the seminorms and the balls $B_n = \{x : d(x,0) < 1/n\}$ form a convex balanced local base for the topology of X.

Theorem 5.5. *X is normable if and only if the origin has a convex bounded neighborhood.*

6 Examples of locally convex spaces

6.1 The spaces $\mathbf{C}(\Omega)$

Let Ω be a nonempty open subset of $\mathbb{K}^n$. We know (e.g. from Differential Geometry) that we can write

$$\Omega = K_1 \cup K_2 \cup \dots$$

where the sets K_i are compact and satisfy $K_i \subset K_{i+1}^{\circ}$.

Now we can topologize the space of continuous functions $f : \Omega \mapsto \mathbb{K}$ by the (separating) family of seminorms

$$p_N(f) = \max_{x \in K_N} |f(x)|$$

by Theorem 5.4. We clearly have $p_1 \leq p_2 \leq \dots$, and therefore the sets

$$V_N = \{f \in C(\Omega) : p_N(f) < 1/N\}$$

form a (countable) local base.

It is easy to see that the metric $d(f,g) = \max_i 2^{-i} \dfrac{p_i(f-g)}{1+p_i(f-g)}$, which is compatible and invariant by Exercise 5.2 is a complete metric; hence this space is a Fréchet space.

By Theorem 5.4, a subset W of $C(\Omega)$ is bounded if there are numbers M_N s.t.
$f \in W \Rightarrow p_N(f) < M_N$.

An open set containing 0 has to contain some V_N, and every V_N contains functions f that make $p_{N+1}(f)$ arbitrary large. Hence, $C(\Omega)$ is not locally bounded. In particular, it is not normable.

6.2 The spaces $\mathbf{C}^\infty(\Omega)$

Let Ω be as above, in particular the union of countably many compact sets K_i.

For a vector $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, we define

$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \dots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$$

Now $C^\infty(\Omega)$ consists of all functions f such that $D^\alpha f$ is continuous for all α.

In a way similar to the one in section 6.1, we get seminorms by

$$p_N(f) = \max\{|D^\alpha f(x)| : x \in K_N, \sum_{i=1}^n \alpha_i \leq N\}$$

Here, a local base is given by

$$V_N = \{f \in C^\infty(\Omega) : p_N(f) < 1/N\}$$

Again, it is not hard to see that this space is actually a Fréchet space (i.e. a complete metric space, when equipped with a compatible, invariant metric).

With some advanced theorems from conventional analysis (see Rudin, p. 35), it is possible to see that $C^\infty(\Omega)$ has the Heine-Borel property (cf. Exercise 4.1), which implies that it is not locally bounded and not normable (because it clearly has infinitely many dimensions).

For suitable Ω, one can consider the subspaces of $C^\infty(\Omega)$ given by the functions whose support is contained in a given compact set K. These spaces are denoted by $\mathcal{D}_K$. It is easy to argue that they are closed subspaces and infinite dimensional, hence they are still not normable by the above.

Vivian has introduced the spaces $W_2^k(\Omega)$, which are normed spaces (the elements are a subset of $C^k(\Omega)$).

The given examples show why one wants to study topological Vector spaces as opposed to normed spaces: We automatically get results applicable to the spaces mentioned above, which are outside the range of normable spaces.

7 The Hahn-Banach Theorems

Lastly, I want to state (some of) the Hahn-Banach Theorems, which are beautiful in their generality. Quoting Rudin: "[They assure] a supply of continuous linear funtionals that is adequate for a highly developed duality theory" (p. 56).

Definition 7.1:
X^* is the *dual space* of X, i.e. the space of **continuous linear functionals** on X.

Theorem 7.1. *Suppose M is a subspace of a real vector space X, a map $p : X \to \mathbb{R}$ satisfies $p(x+y) \leq p(x) + p(y)$ and $p(tx) = tp(x)$ for $x, y \in X$, $t \geq 0$.*

Then for every linear map $f : M \to \mathbb{R}$ with $f(x) \leq p(x)$ on M, there exists a linear $\Lambda : X \to \mathbb{R}$ which extends f and satisfies

$$-p(-x) \leq \Lambda x \leq p(x)$$

A similar theorem can be given for complex vector spaces.

Theorem 7.2. *In a locally convex TVS X, any two (different) points can be separated by an element of X^*.*

Theorem 7.3. *Suppose M is a subspace of a locally convex TVS X; let $x_0 \in X$. If x_0 is not in the closure of M, then there exists $\Lambda \in X^*$ such that $\Lambda x_0 = 1$ and $\Lambda M = 0$.*

Theorem 7.4. *Every continuous linear functional on a subspace M of a locally convex TVS X can be extended to X by an element of X^*.*